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NUMERICAL DETERMINATION OF MINIMUM MASS STRUCTURES

WITH SPECIFIED NATURAL FREQUENCIES

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A. MIELE, A. MANGIAVACCHI, B.P. MOHANTY, and A.K. WU

RICE UNIVERSITY

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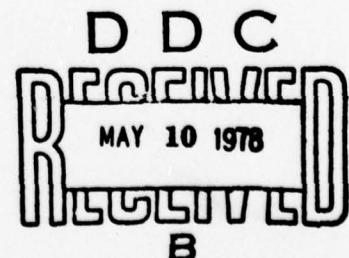A. MIELE, A. MANGIAVACCHI, B.P. MOHANTY, and A.K. WU

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Numerical Determination of Minimum Mass Structures

with Specified Natural Frequencies[1]

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A. MIELE[2], A. MANGIAVACCHI[3], B.P. MOHANTY[4], and A.K. WU[5]

Abstract. The problem of the axial vibration of a cantilever beam is investigated both analytically and numerically. The mass distribution that minimizes the total mass for a given value of the frequency parameter β is determined using both

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the sequential ordinary gradient-restoration algorithm (SOGRA)
and the modified quasilinearization algorithm (MQA). Concern-
ing the minimum value of the mass, SOGRA leads to a solution
precise to at least 4 significant digits and MQA leads to a
solution precise to at least 6 significant digits.

Comparison of the optimal beam (a variable-section beam)
with a reference beam (a constant-section beam) shows that the
weight reduction depends strongly on the frequency parameter β.
This weight reduction is negligible for $\beta \to 0$, is 11.3% for
$\beta = 1$, is 55.3% for $\beta = 1.4$, and approaches 100% for $\beta \to \pi/2$.

Key Words. Structural optimization, dynamic optimization,
axial vibrations, frequency constraint, fundamental frequency
constraint, optimal structures, cantilever beams, bars, sequen-
tial gradient-restoration algorithm, modified quasilinearization
algorithm, numerical methods, computing methods.

1.   Introduction

Under the sponsorship of the Office of Scientific Re-
search, Office of Aerospace Research, United States Air Force,
the Aero-Astronautics Group of Rice University has developed
in recent years several algorithms dealing with the numerical
solution of optimal control problems on a digital computer.

Noteworthy among those algorithms are the sequential
ordinary gradient-restoration algorithm (SOGRA, Refs. 3-4) and
the modified quasilinearization algorithm (MQA, Refs. 5-6).
For a survey of these algorithms, see Ref. 7.

The algorithms discussed in Refs. 3-7 are general, all-
purpose algorithms. As far as aerospace engineering is con-
cerned, these algorithms can be applied to a variety of fields,
for example, optimum trajectories, optimum aerodynamic shapes,
and optimum structures. With reference to optimum structures,
two categories of problems can be identified: (i) static op-
timization problems, for example, minimization of the weight
of a structure for a given load distribution; and (ii) dynamic
optimization problems, for example, minimization of the weight
of a structure for given frequency constraints.

This paper is the first of a series dealing with dynamic
optimization problems and is being written at the suggestion
of Dr. V.B. Venkayya, Flight Dynamics Laboratory, Wright
Patterson AFB, Ohio. The questions posed by Dr. Venkayya were

as follows: can SOGRA and MQA be usefully applied to the
solution of dynamic optimization problems? How reliable and
precise are the solutions obtained with these algorithms under
realistic engineering conditions?

At Dr. Venkayya's urging, the problem of the axial
vibration of a cantilever beam was investigated, with the
following objective in mind: to find the mass distribution
that minimizes the total mass for a prescribed fundamental
frequency. This problem was studied first, because a closed-
form solution, due to Turner (Ref. 8), is available. Hence,
it constitutes an ideally suited test problem for checking
the accuracy and reliability of SOGRA and MQA in the dynamic
optimization of structures.

Outline. This paper is divided in three parts: physi-
cal problem (Sections 2-4), optimization problem (Sections
5-6), and numerical results (Sections 7-9). The first part
contains a description of the physical problem, both in di-
mensional form (Section 2) and in dimensionless form (Section
3); a reference solution, that pertaining to a constant-section
beam, is given in Section 4. The second part contains a des-
cription of the mathematical problem (Section 5); Section 6
presents several alternative formulations and an essential
simplification: the elimination of a redundant constraint.
The third part contains the description of the algorithms
(Section 7), the experimental conditions (Section 8), the
numerical results (Section 9), and the conclusions (Section 10).

Notation.   Throughout the paper, the following notation is employed.

Physical System

A   Cross-sectional area, ft^2

E   Modulus of elasticity, lb ft^{-2}

F_e Elastic force, lb

F_i Inertia force, lb

L   Length of the beam, ft

M   Mass per unit length, lb ft^{-2} sec^2

M_o Tip mass, lb ft^{-1} sec^2

M_* Total mass of the beam, lb ft^{-1} sec^2

t   Time, sec

U   Axial displacement, ft

x   Axial coordinate, ft

Y(x)   Displacement function, ft

Z(t)   Temporal function, dimensionless

ε Normal strain, dimensionless

ρ Density, lb ft^{-4} sec^2

σ Normal stress, lb ft^{-2}

ω Natural frequency, sec^{-1}

Subscripts for the Physical System

x   Derivative with respect to the axial coordinate

t   Derivative with respect to time

Normalized System

I   Total mass of the beam, $I = M_*/M_O$

m   Mass per unit length, $m = ML/M_O$

t   Axial coordinate, $t = x/L$

u   First derivative of the displacement, $u = \dot{y}$

w   Second derivative of the displacement, $w = \ddot{y}$

y   Axial displacement, $y = Y(x)/Y(L)$

z   Auxiliary variable, $z = mu$

β   Frequency parameter, $\beta = \omega L \sqrt{(\rho/E)}$.

Superscript for the Normalized System

Derivative with respect to the axial coordinate t

(for example, $\dot{y} = dy/dt$)

2.   Physical System

We consider the problem of the axial vibration of a cantilever beam or bar, that is, a beam having a fixed end and a free end. Attached to the free end is the concentrated mass M_o . The beam has variable cross section; hence, it has variable mass per unit length M. The total mass of the beam is given by[6]

$$M_* = \int_0^L M dx \ ,\tag{1}$$

where x denotes an axial coordinate, measured from the fixed end of the beam, and L is the length of the beam. Note that $x = 0$ at the fixed end and $x = L$ at the free end.

Assume that the system composed by the masses M_o and M_* is excited by some external axial disturbance. Then, the beam vibrates axially under the combined effect of the elastic forces and the inertia forces. These forces are in equilibrium.

Equilibrium Equation. Let F_e denote the elastic force at station x; let $\tilde{F}_e$ denote the elastic force at station $\tilde{x} = x + dx$; let $\tilde{F}_e - F_e$ denote the net elastic force acting on the beam element having length dx; and let dF_i denote the inertia force associated with the mass element Mdx. The equilibrium condition between elastic forces and inertia forces requires that

$$\tilde{F}_e - F_e = dF_i \ ,\tag{2}$$

---

[6]Note that $M = M(x)$.

where [7]

$$F_e = \sigma A , \tag{3-1}$$

$$\tilde{F}_e = \sigma A + [\partial(\sigma A)/\partial x]dx , \tag{3-2}$$

$$dF_i = M U_{tt} dx . \tag{3-3}$$

In the above equations, σ denotes the normal stress, U the axial displacement, U_t the axial velocity, U_{tt} the axial acceleration, and t is the time. From (2)-(3), the equilibrium condition takes the form

$$\partial(\sigma A)/\partial x = MU_{tt} . \tag{4}$$

The left-hand side of Eq. (4) can be expressed in terms of M and U if two observations are made. First, the mass per unit length M, the cross-sectional area A, and the density of the material ρ satisfy the relation

$$M = \rho A . \tag{5}$$

Second, on account of Hooke's law, the normal stress σ, the normal strain $\epsilon = U_x$, and the modulus of elasticity E satisfy the relation[8]

$$\sigma = E\epsilon = EU_x . \tag{6}$$

Therefore, upon combining Eqs. (4)-(6), the equilibrium condition

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[7] Note that $A = A(x)$.

[8] In Eq. (6), a one-dimensional state of stress is assumed.

takes the form

$$(E/\rho)(\partial/\partial x)(MU_x) - MU_{tt} = 0 . \tag{7}$$

Boundary Conditions. The partial differential equation
(7) must be solved subject to appropriate boundary conditions.

(i) At the fixed end of the beam, the displacement must
vanish. Therefore,

$$U = 0 \quad \text{at} \quad x = 0 . \tag{8}$$

(ii) At the free end of the beam, the elastic force act-
ing on the cross section A must balance the inertia force
associated with the concentrated mass M_o. Clearly,

$$F_e + F_i = 0 \quad \text{at} \quad x = L , \tag{9}$$

where

$$F_e = \sigma A = (E/\rho)MU_x , \tag{10}$$

$$F_i = M_o U_{tt} . \tag{11}$$

Therefore, upon combining Eqs. (9)-(11), we see that

$$(E/\rho)MU_x + M_o U_{tt} = 0 \quad \text{at} \quad x = L . \tag{12}$$

(iii) The spatial boundary conditions (8) and (12) must
be completed by appropriate temporal boundary conditions.

However, these temporal boundary conditions are not specified here: we are only interested in the spatial behavior of the system.

Separation of Variables. We assume that the nature of the temporal boundary conditions is such that the technique of separation of variables is applicable to the system composed of Eqs. (7), (8), (12). Therefore, we write the unknown function $U(x,t)$ in the form

$$U(x,t) = Y(x) Z(t). \tag{13}$$

Upon substitution, we see that the previous differential system splits into a spatial system and a temporal system, each described by ordinary differential equations.

(i) For the spatial system, Eqs. (7), (8), (12) can be rewritten as

$$(E/\rho)(d/dx)(MY_x) + \omega^2 MY = 0 , \tag{14}$$

$$Y = 0 \quad \text{at} \quad x = 0 , \tag{15}$$

$$(E/\rho)MY_x - \omega^2 M_o Y = 0 \quad \text{at} \quad x = L , \tag{16}$$

where ω denotes any of the natural frequencies of the system under consideration.[9] Clearly, the solution $Y(x)$ of (14)-(16) depends on the mass distribution $M(x)$ as well as on several constants characteristic of the system (L,E,ρ,ω,M_o).

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[9] Note that ω^2 is the separation constant, that is, the constant which allows the separation of the spatial system from the temporal system.

(ii) For the temporal system, Eq. (7) implies that

$$z_{tt} + \omega^2 z = 0 \ .$$

(17)

Of course, this equation must be completed by appropriate and consistent temporal boundary conditions.

(iii) In the remainder of this paper, we focus our attention on the solution of the spatial system (14)-(16). For problems where the function $M(x)$ is prescribed a priori (for instance, the study of the axial vibration of a constant section beam), the system (14)-(16) is linear. For problems where the function $M(x)$ is to be determined (for instance, the optimization problem described later on in this paper), the system (14)-(16) is nonlinear. In either case, the system (14)-(16) is homogeneous in the unknown function $Y(x)$, a circumstance to be exploited later on in the paper.

3.   Normalized System

The equations describing the spatial system can be re-written in a simpler form if the following dimensionless quantities are introduced:

$$t = x/L, \qquad m = ML/M_o, \qquad y = Y(x)/Y(L), \qquad (18)$$

$$\beta = \omega L \sqrt{(\rho/E)}, \qquad I = M_*/M_o. \qquad (19)$$

With these definitions, the mass integral (1) becomes

$$I = \int_0^1 m\, dt \qquad (20)$$

and Eqs. (14)-(16) become[10]

$$(d/dt)(m\dot{y}) + \beta^2 m y = 0, \qquad (21)$$

$$y(0) = 0, \qquad (22)$$

$$m(1)\dot{y}(1) = \beta^2, \qquad y(1) = 1. \qquad (23)$$

The quantity β, which is proportional to the natural frequency ω, is called the frequency parameter. The additional boundary condition (23-2) is due to the fact that the displacement function $Y(x)$ has been normalized in terms of $Y(L)$, the value assumed by the displacement function at the free end of the beam. This is possible, owing to the fact that the spatial system (14)-(16) is homogeneous in the displacement function $Y(x)$.

Note that, for problems where the function m(t) is pres-
cribed a priori (for instance, the study of the axial vibration
of a constant section beam), the system (21)-(23) is linear.
On the other hand, for problems where the function m(t) is to
be determined (for instance, the minimum mass problem described
in Section 5), the system (21)-(23) is nonlinear.

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[10]Equation (23-2) is a normalization condition for the dis-
placement function y(t).

4.   Constant-Section Beam

In this section, we consider the solution of the system (21)-(23) for the particular case of a constant-section structure. For

$$m = \text{const} , \tag{24}$$

the mass integral (20) reduces to

$$I = m \tag{25}$$

and Eqs. (21)-(23) become

$$\ddot{y} + \beta^2 y = 0 , \tag{26}$$

$$y(0) = 0 , \tag{27}$$

$$m\dot{y}(1) = \beta^2 , \qquad y(1) = 1 . \tag{28}$$

The solution of (26), (27), and (28-2), valid for any value of the frequency parameter β, is given by

$$y = \sin(\beta t)/\sin\beta . \tag{29}$$

The mass per unit length m corresponding to the frequency parameter β can be computed from (28-1):

$$m = \beta\tan\beta . \tag{30}$$

Therefore, the mass integral is given by

$$I = \beta \tan \beta \; . \tag{31}$$

In order to understand the significance of (31), recall that I is the ratio of the beam mass M_* to the tip mass M_o, and consider two limiting cases: (i) negligible mass ratio and (ii) infinite mass ratio.

If $I = 0$, then the solution of (31) is

$$\beta = n\pi \; , \qquad n = 0, 1, 2, \ldots \; . \tag{32}$$

On the other hand, if $I = \infty$, then the solution of (31) is

$$\beta = (2n + 1)\pi/2 \; , \qquad n = 0, 1, 2, \ldots \; . \tag{33}$$

Since the first natural frequency or fundamental frequency corresponds to $n = 0$, we conclude that, for any mass ratio in the range

$$0 \le I \le \infty \; , \tag{34}$$

the smallest value of the frequency parameter β consistent with the transcendental equation (31) lies in the range

$$0 \le \beta \le \pi/2 \; . \tag{35}$$

We shall recall this result in the optimization problem formulated in Section 5. Specifically, we shall pose the following problem. For a given value of the frequency parameter

β in the range (35), is there a beam yielding a smaller value
of the mass integral (20) than the value (31) associated with
the constant-section beam? In particular, is there a variable-
section design yielding the smallest value of the mass integral
(20) among all the designs satisfying the feasibility equations
(21)-(23)?

5.   Optimization Problem

In the previous section, the solution valid for a
constant-section beam was given. Here, we consider a variable-
section beam and formulate the following optimization problem:
For a given value of the frequency parameter β, find the mass
distribution $m(t)$ and the displacement distribution $y(t)$ such
that the mass integral is minimized, while the feasibility
equations are satisfied. This is a problem of the calculus of
variations or a problem of optimal control, depending on the
formulation being employed (Refs. 9-10).

Formulation (F1).   In this formulation, the optimiza-
tion problem has the following format: [11]

$$I = \int_0^1 m\,dt , \tag{36}$$

$$m\ddot{y} + \dot{m}\dot{y} + \beta^2 m y = 0 , \tag{37}$$

$$y(0) = 0 , \tag{38}$$

$$y(1) = 1 , \qquad m(1)\dot{y}(1) = \beta^2 . \tag{39}$$

Clearly, the unknowns are the functions $m(t)$ and $y(t)$.

Formulation (F2).   Let the following auxiliary varia-
bles be introduced:

$$u = \dot{y} , \qquad w = \ddot{y} . \tag{40}$$

---
[11]The dot denotes derivative with respect to the independent
variable t.

With the aid of these variables, the problem (36)-(39) can be
rewritten in optimal control format as follows:

$$I = \int_0^1 m\,dt \,, \tag{41}$$

$$\dot{y} = u \,, \qquad \dot{m} = -(m/u)(w + \beta^2 y) \,, \qquad \dot{u} = w \,, \tag{42}$$

$$y(0) = 0 \,, \tag{43}$$

$$y(1) = 1 \,, \qquad m(1)u(1) = \beta^2 \,. \tag{44}$$

In this formulation, the unknowns are the state variables $y(t)$,
$m(t)$, $u(t)$ and the control variable $w(t)$. Note that the control
appears linearly in the differential equations. Hence, the
Hamiltonian is linear in the control $w(t)$, and the problem is
singular. Also note that the system (41)-(44) is autonomous,
since the independent variable t does not appear explicitly on
the right-hand side of Eqs. (41)-(42). As a consequence, the
Hamiltonian is constant along the interval of integration.

6.   Alternative Formulations of the Optimization Problem

   In this section, we present some alternative formula-
tions of the optimization problem, having important computa-
tional implications.

   Formulation (F3).   Let the following auxiliary variable
be introduced:

$$z = mu .\qquad\qquad(45)$$

Let the variable $m(t)$ of Formulation (F2)   be eliminated
and replaced by $z(t)$.   With this understanding, the problem
(41)-(44) can be reformulated as follows:

$$I = \int_0^1 (z/u)dt ,\qquad\qquad(46)$$

$$\dot{y} = u , \qquad \dot{z} = -\beta^2 yz/u , \qquad \dot{u} = w ,\qquad\qquad(47)$$

$$y(0) = 0 ,\qquad\qquad(48)$$

$$y(1) = 1 , \qquad z(1) = \beta^2 .\qquad\qquad(49)$$

In this formulation, the unknowns are the state variables $y(t)$,
$z(t)$, $u(t)$ and the control variable $w(t)$.   After these functions
are determined, the function $m(t)$ can be computed a posteriori
with the relation

$$m = z/u .\qquad\qquad(50)$$

A characteristic of this formulation is that all the
boundary conditions are linear. The Hamiltonian is linear in
the control $w(t)$, and hence the problem is singular. Once
more, the system is autonomous, and the Hamiltonian is cons-
tant along the interval of integration.

Formulation (F4).   Inspection of (46)-(49) shows that
the control $w(t)$ appears only in Eq. (47-3). Also, both
$u(0)$ and $u(1)$ are free. Hence, one surmises that the differential
constraint (47-3) might be redundant. This can be shown as follows.
Assume that the problem (46)-(49) is solved bypassing Eq.
(47-3); with the solution $y(t)$, $z(t)$, $u(t)$ known, one can al-
ways determine a posteriori the function $w(t)$ in such a way
that the feasibility equation (47-3) is satisfied. This simple
but important observation leads to a new formulation of the
optimal control problem:

$$I = \int_0^1 (z/u)dt , \tag{51}$$

$$\dot{y} = u , \qquad \dot{z} = -\beta^2 yz/u , \tag{52}$$

$$y(0) = 0 , \tag{53}$$

$$y(1) = 1 , \qquad z(1) = \beta^2 . \tag{54}$$

In this formulation, the unknowns are the state variables $y(t)$,
$z(t)$ and the control variable $u(t)$. After these functions are

determined, the functions m(t) and w(t) are computed a post-eriori with the relations

$$m = z/u , \qquad w = \dot{u} . \tag{55}$$

A characteristic of this formulation is that all the boundary conditions are linear. In addition, the state varia-ble u(t) of Formulation (F3) becomes the control variable u(t) of Formulation (F4). The Hamiltonian is nonlinear in the control u(t), and hence the problem is nonsingular. Once more, the system is autonomous, and the Hamiltonian is constant along the interval of integration.

Formulation (F5). This formulation is a slight modifi-cation of the previous one and is obtained by eliminating the control variable u(t) and replacing it with the new control variable m(t). On account of (55-1), Eqs. (51)-(54) become

$$I = \int_0^1 m\,dt , \tag{56}$$

$$\dot{y} = z/m , \qquad \dot{z} = -\beta^2 ym , \tag{57}$$

$$y(0) = 0 , \tag{58}$$

$$y(1) = 1 , \qquad z(1) = \beta^2 . \tag{59}$$

In this formulation, the unknowns are the state variables y(t), z(t) and the control variable m(t). After these functions are

determined, the functions $u(t)$ and $w(t)$ are computed a post-
eriori with the relations

$$u = z/m, \qquad w = \dot{u} . \tag{60}$$

A characteristic of this formulation is that all the
boundary conditions are linear. In addition, the Hamiltonian
is nonlinear in the control $m(t)$, and hence the problem is non-
singular. Once more, the system is autonomous, and the Hamil-
tonian is constant along the interval of integration. Note
that this formulation can be obtained directly from (20)-(23)
after consideration of the relation

$$z = m\dot{y} . \tag{61}$$

Remark. The analytical justification for the passage
from Formulation (F3) to Formulation (F4) is as
follows. With reference to Formulation (F3),   let $\lambda_1(t)$,
$\lambda_2(t)$, $\lambda_3(t)$ denote variable Lagrange multipliers associated
with the differential constraints (47). Let

$$H = z/u - \lambda_1 u + \lambda_2 \beta^2 yz/u - \lambda_3 w \tag{62}$$

denote the Hamiltonian of the system (46)-(49). The condition
optimizing the control $w(t)$ of Formulation (F3)   takes the

form

$$H_w = 0 , \qquad 0 \leq t \leq 1 , \tag{63}$$

which implies that

$$\lambda_3(t) = 0 , \qquad 0 \leq t \leq 1 . \tag{64}$$

Note that both $u(0)$ and $u(1)$ are free; hence, the transversality condition requires that

$$\lambda_3(0) = \lambda_3(1) = 0 , \tag{65}$$

and these relations are consistent with (64). From (64)-(65), it is natural to infer that the differential constraint (47-3) is redundant, a circumstance which has important analytical and computational implications.

7.   Description of the Algorithms

With the sequential ordinary gradient-restoration al-
gorithm (SOGRA, Refs. 3-4), numerical solutions can be obtained
using any of the optimal control formulations of Sections 5-6.
This is because SOGRA can be applied regardless of whether the
problem is singular or nonsingular. On the other hand, with
the modified quasilinearization algorithm (MQA, Refs. 5-6),
numerical solutions can be obtained using Formulations (F4)
and (F5), but not Formulations (F2) and (F3). This
is because the optimal control problem is singular with the
latter formulations, while it is nonsingular with the former.
For these reasons, the experiments reported here refer to
Formulations (F4) and (F5).

Note that both SOGRA and MQA require the state vector to be
known at the initial point, so that the trajectories in the
state space can be anchored at the initial point. With refer-
ence to the fourth and fifth formulations (state vector given
at the final point), this situation can be achieved by re-
placing the independent variable t with its complement

$$\tau = 1-t , \tag{66}$$

that is, by reversing the sense of integration and inter-
changing the final point with the initial point. Even though
this replacement of independent variable is necessary

computationally, it is not formally executed here, for two
reasons: for the sake of brevity and in order to avoid compli-
cating the description of the problems and the interpretation
of the results. Therefore, in the following sections, we
shall retain the independent variable t, that is, we shall
imagine the trajectories in the state space to be anchored at
the final point at every iteration.

Formulation (F4).   With the formulation represented by
Eqs. (51)-(54), the Hamiltonian is given by[12]

$$H = z/u - \lambda_1 u + \lambda_2 \beta^2 yz/u \, , \tag{67}$$

where $\lambda_1(t)$ and $\lambda_2(t)$ denote variable Lagrange multipliers
associated with the differential constraints (52). Hence, the
optimality conditions take the form[12]

$$\lambda_1 + (z/u^2)(1 + \lambda_2 \beta^2 y) = 0 \, , \tag{68}$$

$$\dot{\lambda}_1 = \lambda_2 \beta^2 z/u \, , \qquad \dot{\lambda}_2 = (1/u)(1 + \lambda_2 \beta^2 y) \, , \tag{69}$$

$$\lambda_1(0) + \mu = 0 \, , \qquad \lambda_2(0) = 0 \, . \tag{70}$$

---

[12] Equation (68) optimizes the control distribution u(t). In Eq.
(70-1), the symbol μ denotes a constant Lagrange multiplier
associated with the initial condition (53).

Formulation (F5).   With the formulation represented

by Eqs. (56)-(59), the Hamiltonian is given by

$$H = m - \lambda_1 z/m + \lambda_2 \beta^2 ym ,\tag{71}$$

where $\lambda_1(t)$ and $\lambda_2(t)$ denote variable Lagrange multipliers

associated with the differential constraints (57).   Hence, the

optimality conditions take the form[13]

$$\lambda_1 z/m^2 + 1 + \lambda_2 \beta^2 y = 0 ,\tag{72}$$

$$\dot{\lambda}_1 = \lambda_2 \beta^2 m , \qquad \dot{\lambda}_2 = -\lambda_1/m ,\tag{73}$$

$$\lambda_1(0) + \mu = 0 , \qquad \lambda_2(0) = 0 .\tag{74}$$

Auxiliary Functionals.   In addition to the functional I,

several auxiliary functionals are of interest in the implemen-

tation of SOGRA and MQA.   These auxiliary functionals are

denoted by the symbols J, P, Q, R and have the following mean-

ing: J is the augmented functional, that is, the functional I

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[13] Equation (72) optimizes the control distribution $m(t)$.   In
Eq. (74-1), the symbol μ denotes a constant Lagrange multi-
plier associated with the initial condition (58).

augmented linearly by the constraints through Lagrange multi-
pliers; P is the constraint error, that is, the norm squared
of the error in the feasibility equations; Q is the optimality
condition error, that is, the norm squared of the error in the
optimality conditions; and $R = P + Q$ is the total error in the
system composed of the feasibility equations and the optimali-
ty conditions. In the definitions of J, P, Q, R, it is tacitly
assumed that the final conditions (54) or (59) are satisfied at
every iteration of SOGRA and MQA.

Convergence Conditions. The auxiliary functionals P, Q,
R are defined in such a way that

$$P = Q = R = 0 \tag{75}$$

for the optimal solution. For an approximation to the optimal
solution, Eqs. (75) are replaced by the inequalities

$$P \leq \varepsilon_1 , \qquad Q \leq \varepsilon_2 , \tag{76}$$

or

$$R \leq \varepsilon_3 , \tag{77}$$

where ε_1, ε_2, ε_3 are small, preselected numbers. Inequalities
(76) are of interest for SOGRA, and Ineq. (77) is of interest
for MQA.

Sequential Ordinary Gradient-Restoration Algorithm. The
sequential ordinary gradient-restoration algorithm (SOGRA,
Refs. 3-4) is an iterative technique which includes a sequence
of cycles having the following properties: (i) the functions
available both at the beginning and at the end of each cycle
are feasible; that is, they are consistent with the feasibi-
lity equations within the preselected accuracy (76-1); and (ii)
the functions produced at the end of each cycle are charac-
terized by a value of the functional I which is smaller than
that associated with the functions available at the beginning
of the cycle.

To achieve the above properties, each cycle is made of
two phases, a gradient phase and a restoration phase. These
phases are now described.

The gradient phase is started only when the constraint
error P satisfies Ineq. (76-1). It involves a single iter-
ation, which is designed to decrease the value of the func-
tional I or the augmented functional J, while satisfying the
constraints to first order. During this iteration, the first
variation of the functional I is minimized, subject to the
linearized constraints and a quadratic constraint on the vari-
ations of the control.

The restoration phase is started only when the constraint
error P violates Ineq. (76-1). The restoration phase involves

one or more iterations. In each restorative iteration, the
objective is to reduce the constraint error P, while the con-
straints are satisfied to first order and the norm of the
variations of the control is minimized. The restoration phase
is terminated whenever Ineq. (76-1) is satisfied.

In summary, the main properties of the sequential ordi-
nary gradient-restoration algorithm can be written as follows:

$$\tilde{J} < J \, , \qquad P \leq \varepsilon_1 \, , \tag{78}$$

$$\tilde{P} < P \, , \qquad P > \varepsilon_1 \, , \tag{79}$$

$$\tilde{I} < I \, , \qquad P \leq \varepsilon_1 \, , \qquad \tilde{P} \leq \varepsilon_1 \, , \tag{80}$$

where (78) hold for a gradient iteration, (79) hold for a
restorative iteration, and (80) hold for a complete gradient-
restoration cycle. In the above relations, I, J, P denote
quantities evaluated at the beginning of an iteration or a
cycle, while $\tilde{I}$, $\tilde{J}$, $\tilde{P}$ denote quantities evaluated at the end
of an iteration or a cycle.

At each iteration of the gradient phase or the restora-
tion phase, a linear, two-point boundary-value problem must be
solved. If the method of particular solutions is employed
(Refs. 11-12), one must execute $q + 1$ independent sweeps of the
linearized system governing the variations associated with the

gradient phase or the restoration phase, where q is the number
of initial conditions. Since $q = 1$ for the present problem, each
gradient iteration or restorative iteration requires 2 sweeps
of the linearized system. For the details, see Refs. 3-4 and
11-12.

Modified Quasilinearization Algorithm. The modified
quasilinearization algorithm (MQA, Refs. 5-6) is an iterative
technique which includes a sequence of iterations having the
following property: the functions produced at the end of each
iteration are characterized by a value of the functional R
which is smaller than that associated with the functions avail-
able at the beginning of the iteration. Therefore, the main
property of the modified quasilinearization algorithm can be
written as follows:

$$\tilde{R} < R, \tag{81}$$

with the implication that

$$\tilde{P} < P \quad \text{and/or} \quad \tilde{Q} < Q. \tag{82}$$

At each iteration, both the feasibility equations and
the optimality conditions are satisfied to first order. To
achieve this, a linear, two-point boundary-value problem must
be solved. If the method of particular solutions is employed
(Refs. 11-12), one must execute $n + 1$ independent sweeps of the

linearized system governing the variations associated with
MQA, where n is the number of state variables. Since n = 2 for
the present problem, each iteration requires 3 sweeps of the
linearized system. For the details, see Refs. 5-6 and 11-12.

8.   Experimental Conditions

The problem of determining the mass distribution that
minimizes the total mass for a given value of the frequency
parameter β was solved using both Formulation (F4) and
Formulation (F5). The following values were given to the
frequency parameter:

$$\beta = \pi/6 , \qquad \beta = \pi/5 , \qquad \beta = \pi/4 , \qquad \beta = \pi/3 , \qquad \beta = \pi/2 . \quad (83)$$

Computations were performed on the IBM 370/155 computer
of Rice University, Houston, Texas, using both the sequential
ordinary gradient-restoration algorithm (SOGRA) and the modi-
fied quasilinearization algorithm (MQA). Both algorithms were
programmed in FORTRAN IV, and the numerical results were ob-
tained in double-precision arithmetic.

The interval of integration was divided into 100 steps
for SOGRA and 50 steps for MQA. The differential equations were
integrated using Hamming's modified predictor-corrector method,
with a special Runge-Kutta starting procedure. The definite inte-
grals I, J, P, Q were computed using a modified Simpson's
rule.

Nominal Functions. For each formulation, two sets of
nominal functions were employed, more specifically:

$$(N1) \quad y(t) = t , \qquad z(t) = \beta^2 , \qquad u(t) = 1 , \qquad (84)$$

(N2) $y(t) = t^2$, $z(t) = \beta^2$, $u(t) = 1$,   (85)

and

(N3) $y(t) = t$, $z(t) = \beta^2$, $m(t) = 1$,   (86)

(N4) $y(t) = t^2$, $z(t) = \beta^2$, $m(t) = 1$.   (87)

The nominal functions (84) and (85) were employed in connection with Formulation (F4),   and the nominal functions (86) and (87) were employed in connection with Formulation (F5).   Note that these nominal functions satisfy the boundary conditions, but violate the differential constraints.

Stopping Conditions. The sequential ordinary gradient-restoration algorithm was programmed to stop whenever a solution consistent with the following inequalities was obtained:[12]

$$P \leq E-08 , \qquad Q \leq E-04 . \tag{88}$$

The modified quasilinearization algorithm was programmed to stop whenever the following inequality was satisfied:

$$R \leq E-10 , \tag{89}$$

---

[12]The symbol $E \pm ab$ stands for $10^{\pm ab}$.

with the implication that

$$P \leq E\text{-}10 , \qquad Q \leq E\text{-}10 . \tag{90}$$

Remark 8.1. For the sake of brevity, the computational details pertaining to SOGRA and MQA (for instance, solution of the linear, two-point boundary value problem, computation of the stepsize, safeguards, and nonconvergence conditions) are omitted, and the reader is referred to Refs. 3-4 and Refs. 5-6.

Remark 8.2. Ordinarily, algorithms of the second-order type require the specification of nominal functions not only for the state and the control, but also for the multipliers. This is not the case with MQA. Once the nominal state and the nominal control are specified, the nominal multipliers $\lambda_1(t)$, $\lambda_2(t)$, μ are determined automatically through a subroutine which supplies the solution of the following auxiliary minimization problem: minimize the error in the optimality conditions Q with respect to multipliers $\lambda_1(t)$, $\lambda_2(t)$, μ. This is an important feature of MQA, certainly very useful to engineers, since it circumvents the need for an arbitrary guess of the multiplier functions.

9.   Numerical Results

Under the experimental conditions outlined in the pre-
vious section, 40 computer runs were executed: the minimization
problem was solved for five different values of the frequency
parameter, using two algorithms (SOGRA and MQA), two formula-
tions [(F4) and (F5)], and two sets of nominal functions for
each formulation [(N1) and (N2) for Formulation (F4), and (N3)
and (N4) for Formulation (F5)]. Convergence to the desired
stopping condition was achieved in 37 runs. The nonconvergence
occurring in the remaining 3 runs was due to poor choice of the
nominal functions. Incidentally, all the nonconvergence cases
occurred for $\beta = \pi/2$, which is the upper limit to the range of
values of the frequency parameter investigated. The numerical
results are presented in Tables 1-24.

Tables 1-4 present summary results pertaining to SOGRA
at convergence. For each value of the frequency parameter β,
the tables show the number of iterations for convergence N,
the computed value of the functional I, the exact value of the
functional I_e, the number of correct significant digits M
(determined by comparing I with I_e), and the error in the
optimality conditions Q. Clearly, as far as the minimum mass
is concerned, the solutions obtained are precise to at least
4 significant digits; in some cases, they are precise to 6
significant digits.

Tables 5-8 present summary results pertaining to MQA at convergence. For each value of the frequency parameter β, the tables show the number of iterations for convergence N, the computed value of the functional I, the exact value of the functional I_e, the number of correct significant digits M (determined by comparing I with I_e), and the total error in the system R. Clearly, as far as the minimum mass is concerned, the solutions obtained are precise to at least 6 significant digits; in many cases, they are precise to 7 significant digits.

Due to the large number of runs performed, it is impractical to present the convergence history for every value of the frequency parameter β. However, for a particular value of the frequency parameter (namely, $\beta = \pi/4$), the convergence history of SOGRA is given in Tables 9-12 and the convergence history of MQA is given in Tables 13-16. The descent property in I, characteristic of SOGRA, is apparent from Tables 9-12. Analogously, the descent property in R, characterisitc of MQA, is apparent from Tables 13-16.

Due to the large number of runs performed, it is impractical to present the converged solutions $y(t)$, $z(t)$, $m(t)$, $u(t)$, $w(t)$ for every case. Therefore, Tables 17-21 present only one set of converged solutions: these are the solutions corresponding to the lowest value of R achieved computationally for

any given value of the frequency parameter β. Even though
the tolerance level set for MQA is $R \leq E-10$, values of the
total error much lower that $E-10$ were achieved during compu-
tation, more precisely, $R \leq E-16$ for $\beta = \pi/5$ and $R \leq E-18$ for
the remaining values of β.

Table 22 is obtained from Tables 17-21 by normalizing
the mass distribution $m(t)$ with respect to the mass per unit
length at $t = 0$. Therefore, Table 22 shows the ratio $m(t)/m(0)$
for several values of the frequency parameter β. Clearly,
the mass distribution of the optimal beam approaches that of
the constant-section beam for $\beta \to 0$ and deviates considerably
from it for $\beta \to \pi/2$. In particular, the ratio $m(1)/m(0)$ of
tip mass per unit length to root mass per unit length approa-
ches 1 for $\beta \to 0$ and $1/6$ for $\beta \to \pi/2$.

Finally, Tables 23-24 show the mass integral for the
optimal beam I_o, the mass integral for the constant-section
beam I_c, as well as the quantities I_o/I_c and $(I_c - I_o)/I_c$ for
several values of the frequency parameter β. For $\beta \to 0$, the
optimal beam exhibits no weight advantage over the constant-
section beam. For $\beta = 0.5$, the weight advantage is only 0.6%.
For $\beta = 1.0$, the weight advantage is 11.3%. For $\beta = 1.4$, the
weight advantage is 55.3%. Finally, for $\beta \to \pi/2$, the weight
advantage approaches 100%. The explanation for this limiting
result is simple: for $\beta \to \pi/2$, the mass of the optimal beam is
finite, while the mass of the constant-section beam is infi-
nitely large.

Table 1.   Summary results, SOGRA,
           Formulation (F4), Nominal (N1).

β	N	I	I_e	M	Q
$\pi/6$	3	0.3001435E+00	0.3001434E+00	6	0.85E-07
$\pi/5$	3	0.4495492E+00	0.4495488E+00	5	0.85E-06
$\pi/4$	3	0.7545937E+00	0.7545892E+00	4	0.15E-04
$\pi/3$	4	0.1560903E+01	0.1560918E+01	5	0.28E-04
$\pi/2$	10	0.5295951E+01	0.5295977E+01	5	0.88E-03 (*)

(*) Algorithm unable to reach desired stopping condition;
    loss of descent property on I due to numerical inaccuracy.

Table 2.   Summary results, SOGRA,
           Formulation (F4), Nominal (N2).

β	N	I	I_e	M	Q
$\pi/6$	4	0.3001435E+00	0.3001434E+00	6	0.85E-07
$\pi/5$	4	0.4495492E+00	0.4495488E+00	5	0.85E-06
$\pi/4$	4	0.7545937E+00	0.7545892E+00	4	0.15E-04
$\pi/3$	5	0.1560903E+01	0.1560918E+01	5	0.28E-04
$\pi/2$	11	0.5295951E+01	0.5295977E+01	5	0.88E-03(*)

(*) Algorithm unable to reach desired stopping condition;
    loss of descent property on I due to numerical inaccuracy.

Table 3. Summary results, SOGRA,
          Formulation (F5), Nominal (N3).

β	N	I	I_e	M	Q
$\pi/6$	19	0.3001453E+00	0.3001434E+00	5	0.32E-04
$\pi/5$	14	0.4495573E+00	0.4495488E+00	4	0.84E-04
$\pi/4$	10	0.7545631E+00	0.7545892E+00	4	0.36E-04
$\pi/3$	12	0.1560911E+01	0.1560918E+01	6	0.51E-04
$\pi/2$	(*)	(*)	0.5295977E+01	(*)	(*)

(*) Nonconvergence, failure of initial restoration phase;
     algorithm unable to find a feasible solution in less
     than 10 restorative iterations.

Table 4. Summary results, SOGRA,
          Formulation (F5), Nominal(N4).

β	N	I	I_e	M	Q
$\pi/6$	21	0.3001449E+00	0.3001434E+00	5	0.25E-04
$\pi/5$	14	0.4495525E+00	0.4495488E+00	4	0.38E-04
$\pi/4$	10	0.7545679E+00	0.7545892E+00	4	0.26E-04
$\pi/3$	15	0.1560913E+01	0.1560918E+01	6	0.54E-04
$\pi/2$	(*)	(*)	0.5295977E+01	(*)	(*)

(*) Nonconvergence, failure of initial restoration phase;
     algorithm unable to find a feasible solution in less
     than 10 restorative iterations.

Table 5.   Summary results, MQA,
           Formulation (F4), Nominal (N1).

β	N	I	I_e	M	R
$\pi/6$	3	0.3001434E+00	0.3001434E+00	7	0.20E-18
$\pi/5$	3	0.4495488E+00	0.4495488E+00	7	0.74E-16
$\pi/4$	3	0.7545892E+00	0.7545892E+00	7	0.12E-12
$\pi/3$	4	0.1560918E+01	0.1560918E+01	7	0.32E-18
$\pi/2$	(*)	(*)	0.5295977E+01	(*)	(*)

(*) Nonconvergence, stepsize bisection limit reached.

Table 6.   Summary results, MQA,
           Formulation (F4), Nominal (N2).

β	N	I	I_e	M	R
$\pi/6$	3	0.3001434E+00	0.3001434E+00	7	0.17E-18
$\pi/5$	3	0.4495488E+00	0.4495488E+00	7	0.70E-16
$\pi/4$	3	0.7545892E+00	0.7545892E+00	7	0.97E-13
$\pi/3$	4	0.1560918E+01	0.1560918E+01	7	0.11E-18
$\pi/2$	6	0.5295977E+01	0.5295977E+01	7	0.15E-15

Table 7.   Summary results, MQA,
Formulation (F5), Nominal (N3).

β	N	I	I_e	M	R
$\pi/6$	9	0.3001433E+00	0.3001434E+00	6	0.79E-10
$\pi/5$	6	0.4495488E+00	0.4495488E+00	7	0.15E-14
$\pi/4$	5	0.7545892E+00	0.7545892E+00	7	0.41E-18
$\pi/3$	4	0.1560917E+01	0.1560918E+01	6	0.20E-10
$\pi/2$	7	0.5295976E+01	0.5295977E+01	6	0.38E-17

Table 8.   Summary results, MQA,
Formulation (F5), Nominal (N4).

β	N	I	I_e	M	R
$\pi/6$	9	0.3001433E+00	0.3001434E+00	6	0.59E-10
$\pi/5$	7	0.4495487E+00	0.4495488E+00	6	0.55E-10
$\pi/4$	4	0.7545890E+00	0.7545892E+00	6	0.88E-11
$\pi/3$	4	0.1560917E+01	0.1560918E+01	6	0.83E-10
$\pi/2$	7	0.5295976E+01	0.5295977E+01	6	0.78E-18

Table 9.   Convergence history, $\beta = \pi/4$, SOGRA,
           Formulation (F4), Nominal (N1).

N	Phase	P	Q	I
0		0.48E-01		
1	REST	0.10E-33	0.21E-01	0.76081
2	GRAD	0.13E-04		
3	REST	0.82E-33	0.15E-04	0.75459

Table 10. Convergence history, $\beta = \pi/4$, SOGRA,
           Formulation (F4), Nominal (N2).

N	Phase	P	Q	I
0		0.36E+00		
1	REST	0.29E-03		
2	REST	0.90E-33	0.21E-01	0.76081
3	GRAD	0.13E-04		
4	REST	0.27E-32	0.15E-04	0.75459

Table 11.   Convervenge history, $\beta = \pi/4$, SOGRA,
            Formulation (F5), Nominal (N3).

N	Phase	P	Q	I
0		0.27E+00		
1	REST	0.13E+00		
2	REST	0.11E-01		
3	REST	0.58E-04		
4	REST	0.37E-08	0.31E+00	0.83009
5	GRAD	0.16E-01		
6	REST	0.80E-04		
7	REST	0.41E-08	0.46E-02	0.75504
8	GRAD	0.25E-05		
9	REST	0.26E-12	0.24E-03	0.75462
10	GRAD	0.95E-08	0.36E-04	0.75456

Table 12.   Convergence history, $\beta = \pi/4$, SOGRA,
            Formulation (F5), Nominal (N4).

N	Phase	P	Q	I
0		0.55E+00		
1	REST	0.81E-01		
2	REST	0.38E-02		
3	REST	0.92E-05		
4	REST	0.10E-09	0.28E+00	0.82212
5	GRAD	0.13E-01		
6	REST	0.48E-04		
7	REST	0.15E-08	0.46E-02	0.75503
8	GRAD	0.27E-05		
9	REST	0.20E-12	0.19E-03	0.75461
10	GRAD	0.56E-08	0.26E-04	0.75456

Table 13. Convergence history, $\beta = \pi/4$, MQA,
           Formulation (F4), Nominal (N1).

N	P	Q	R	I
0	0.48E-01	0.32E-01	0.80E-01	0.61685
1	0.84E-05	0.23E-02	0.23E-02	0.75463
2	0.48E-08	0.96E-06	0.96E-06	0.75455
3	0.44E-15	0.12E-12	0.12E-12	0.75458

Table 14. Convergence history, $\beta = \pi/4$, MQA,
           Formulation (F4), Nominal (N2).

N	P	Q	R	I
0	0.36E+00	0.27E-01	0.39E+00	0.61685
1	0.27E-03	0.30E-02	0.33E-02	0.74916
2	0.72E-08	0.75E-06	0.76E-06	0.75454
3	0.10E-14	0.96E-13	0.97E-13	0.75458

Table 15.   Convergence history, $\beta = \pi/4$, MQA,
Formulation (F5), Nominal (N3).

N	P	Q	R	I
0	0.27E+00	0.20E+00	0.47E+00	1.00000
1	0.48E-01	0.94E-01	0.14E+00	0.75124
2	0.30E-03	0.71E-02	0.74E-02	0.74933
3	0.69E-06	0.26E-04	0.27E-04	0.75419
4	0.21E-10	0.58E-09	0.61E-09	0.75458
5	0.15E-19	0.39E-18	0.41E-18	0.75458

Table 16.   Convergence history, $\beta = \pi/4$, MQA,
Formulation (F5), Nominal (N4).

N	P	Q	R	I
0	0.55E+00	0.16E+00	0.72E+00	1.00000
1	0.98E-01	0.45E-01	0.14E+00	0.77369
2	0.21E-03	0.19E-02	0.22E-02	0.74914
3	0.14E-06	0.29E-05	0.30E-05	0.75442
4	0.33E-12	0.84E-11	0.88E-11	0.75458

Table 17. Converged solution, $\beta = \pi/6$, MQA.

t	y	z	m	u	w
0.0	0.0000	0.3126	0.3270	0.9557	0.0000
0.1	0.0956	0.3121	0.3261	0.9570	0.0262
0.2	0.1914	0.3108	0.3235	0.9609	0.0524
0.3	0.2878	0.3087	0.3191	0.9675	0.0789
0.4	0.3850	0.3058	0.3131	0.9767	0.1055
0.5	0.4833	0.3021	0.3056	0.9886	0.1325
0.6	0.5829	0.2977	0.2968	1.0032	0.1598
0.7	0.6840	0.2927	0.2868	1.0206	0.1875
0.8	0.7871	0.2870	0.2757	1.0408	0.2157
0.9	0.8923	0.2808	0.2639	1.0638	0.2446
1.0	1.0000	0.2741	0.2515	1.0897	0.2741

Table 18.   Converged solution, $\beta = \pi/5$, MQA.

t	y	z	m	u	w
0.0	0.0000	0.4753	0.5072	0.9371	0.0000
0.1	0.0937	0.4743	0.5052	0.9389	0.0370
0.2	0.1879	0.4715	0.4992	0.9445	0.0741
0.3	0.2828	0.4669	0.4896	0.9538	0.1116
0.4	0.3788	0.4606	0.4764	0.9668	0.1495
0.5	0.4763	0.4527	0.4602	0.9837	0.1880
0.6	0.5756	0.4434	0.4414	1.0044	0.2272
0.7	0.6773	0.4327	0.4204	1.0292	0.2674
0.8	0.7816	0.4209	0.3979	1.0580	0.3085
0.9	0.8890	0.4082	0.3742	1.0909	0.3509
1.0	1.0000	0.3947	0.3499	1.1282	0.3947

Table 19.   Converged solution, $\beta = \pi/4$, MQA.

t	y	z	m	u	w
0.0	0.0000	0.8170	0.9037	0.9041	0.0000
0.1	0.0905	0.8145	0.8981	0.9069	0.0558
0.2	0.1815	0.8071	0.8817	0.9153	0.1120
0.3	0.2737	0.7949	0.8553	0.9293	0.1688
0.4	0.3676	0.7783	0.8200	0.9491	0.2267
0.5	0.4637	0.7578	0.7775	0.9747	0.2860
0.6	0.5627	0.7340	0.7293	1.0063	0.3471
0.7	0.6652	0.7074	0.6774	1.0442	0.4103
0.8	0.7718	0.6786	0.6234	1.0885	0.4761
0.9	0.8832	0.6482	0.5688	1.1395	0.5448
1.0	1.0000	0.6168	0.5150	1.1976	0.6168

Table 20.   Converged solution, $\beta = \pi/3$, MQA.

t	y	z	m	u	w
0.0	0.0000	1.7549	2.0937	0.8381	0.0000
0.1	0.0839	1.7453	2.0709	0.8427	0.0920
0.2	0.1688	1.7171	2.0044	0.8566	0.1851
0.3	0.2556	1.6717	1.8999	0.8798	0.2803
0.4	0.3451	1.6114	1.7654	0.9127	0.3785
0.5	0.4385	1.5390	1.6103	0.9557	0.4808
0.6	0.5366	1.4576	1.4443	1.0091	0.5885
0.7	0.6407	1.3700	1.2760	1.0736	0.7026
0.8	0.7517	1.2791	1.1123	1.1499	0.8244
0.9	0.8711	1.1873	0.9584	1.2388	0.9552
1.0	1.0000	1.0966	0.8175	1.3413	1.0966

Table 21. Converged solution, $\beta = \pi/2$, MQA.

t	y	z	m	u	w
0.0	0.0000	6.1911	9.0703	0.6825	0.0000
0.1	0.0685	6.1155	8.8501	0.6910	0.1691
0.2	0.1387	5.8977	8.2309	0.7165	0.3424
0.3	0.2124	5.5620	7.3207	0.7597	0.5241
0.4	0.2913	5.1422	6.2573	0.8217	0.7188
0.5	0.3774	4.6739	5.1695	0.9041	0.9313
0.6	0.4729	4.1888	4.1521	1.0088	1.1668
0.7	0.5800	3.7118	3.2603	1.1384	1.4312
0.8	0.7015	3.2600	2.5149	1.2962	1.7310
0.9	0.8403	2.8436	1.9134	1.4861	2.0735
1.0	1.0000	2.4674	1.4406	1.7126	2.4674

Table 22.   Optimal mass distribution $m(t)/m(0)$ for several
values of the frequency parameter β .

t	$\beta = 0$	$\beta = \pi/6$	$\beta = \pi/5$	$\beta = \pi/4$	$\beta = \pi/3$	$\beta = \pi/2$
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.0000	0.9972	0.9960	0.9938	0.9891	0.9757
0.2	1.0000	0.9891	0.9843	0.9757	0.9573	0.9074
0.3	1.0000	0.9757	0.9652	0.9464	0.9074	0.8071
0.4	1.0000	0.9573	0.9394	0.9074	0.8431	0.6898
0.5	1.0000	0.9344	0.9074	0.8603	0.7691	0.5699
0.6	1.0000	0.9074	0.8703	0.8071	0.6898	0.4577
0.7	1.0000	0.8768	0.8290	0.7496	0.6094	0.3594
0.8	1.0000	0.8431	0.7845	0.6898	0.5312	0.2772
0.9	1.0000	0.8071	0.7378	0.6294	0.4577	0.2109
1.0	1.0000	0.7691	0.6898	0.5699	0.3904	0.1588

Table 23.   Comparison of the optimal beam (subscript o)
            with the constant section beam (subscript c).

β	I_o	I_c	I_o/I_c	$(I_c-I_o)/I_c$
0	0.00000	0.00000	1.0000	0.0000
$\pi/6$	0.30014	0.30229	0.9929	0.0071
$\pi/5$	0.44954	0.45650	0.9848	0.0152
$\pi/4$	0.75458	0.78539	0.9608	0.0392
$\pi/3$	1.56091	1.81379	0.8606	0.1394
$\pi/2$	5.29597	∞	0.0000	1.0000

Table 24.   Comparison of the optimal beam (subscript o)
            with the constant section beam (subscript c).

β	I_o	I_c	I_o/I_c	$(I_c-I_o)/I_c$
0.0	0.0000	0.0000	1.0000	0.0000
0.1	0.0100	0.0100	1.0000	0.0000
0.2	0.0405	0.0405	0.9999	0.0001
0.3	0.0927	0.0928	0.9992	0.0008
0.4	0.1687	0.1691	0.9976	0.0024
0.5	0.2715	0.2731	0.9941	0.0059
0.6	0.4053	0.4104	0.9874	0.0126
0.7	0.5754	0.5896	0.9760	0.0240
0.8	0.7887	0.8237	0.9575	0.0425
0.9	1.0537	1.1341	0.9291	0.0709
1.0	1.3810	1.5574	0.8868	0.1132
1.1	1.7839	2.1612	0.8254	0.1746
1.2	2.2784	3.0865	0.7382	0.2618
1.3	2.8845	4.6827	0.6160	0.3840
1.4	3.6263	8.1170	0.4468	0.5532
1.5	4.5338	21.1521	0.2143	0.7857
$\pi/2$	5.2959	∞	0.0000	1.0000

10.   Conclusions

In this report, we have investigated the axial vibra-
tion of a cantilever beam both analytically and numerically.
We have determined the mass distribution that minimizes the
total mass for a given value of the fundamental frequency
using both the sequential ordinary gradient-restoration al-
gorithm (SOGRA) and the modified-quasilinearization algorithm
(MQA). Concerning the minimum value of the mass, SOGRA leads
to a solution precise to at least 4 significant digits and
MQA leads to a solution precise to at least 6 significant
digits.

Comparison of the optimal beam (a variable-section beam)
with a reference beam (a constant-section beam) shows that the
weight reduction depends strongly on the frequency parameter β.
This weight reduction is negligible for $\beta \to 0$, is 0.6% for
$\beta = 0.5$, is 11.3% for $\beta = 1$, is 55.3% for $\beta = 1.4$, and approaches
100% for $\beta \to \pi/2$.

11.   Appendix: Analytical Solutions

   In this appendix, we summarize the analytical solutions available in two limiting cases: the constant-section beam and the optimal beam.

   Constant-Section Beam.   For this structure, the functions $y(t)$, $z(t)$, $m(t)$, $u(t)$, $w(t)$ are given by

$$y = \sin(\beta t)/\sin\beta \, , \tag{91-1}$$

$$z = \beta^2 \cos(\beta t)/\cos\beta \, , \tag{91-2}$$

$$m = \beta \tan\beta \, , \tag{91-3}$$

$$u = \beta \cos(\beta t)/\sin\beta \, , \tag{91-4}$$

$$w = -\beta^2 \sin(\beta t)/\sin\beta \, . \tag{91-5}$$

The associated mass integral is given by

$$I = \beta \tan\beta \, . \tag{92}$$

   Optimal Beam.   For this structure, the functions $y(t)$, $z(t)$, $m(t)$, $u(t)$, $w(t)$ are given by (Ref. 8)

$$y = \sinh(\beta t)/\sinh(\beta) \, , \tag{93-1}$$

$$z = \beta^2 \cosh(\beta)/\cosh(\beta t) \, , \tag{93-2}$$

$$m = \beta \sinh(\beta)\cosh(\beta)/\cosh^2(\beta t) \, , \tag{93-3}$$

$$u = \beta\cosh(\beta t)/\sinh(\beta) , \tag{93-4}$$

$$w = \beta^2\sinh(\beta t)/\sinh(\beta) . \tag{93-5}$$

The associated mass integral is given by

$$I = \sinh^2(\beta) . \tag{94}$$

Limiting Case. For $\beta = 0$, Eqs. (91) and (93) reduce to

$$y = t , \quad z = 0 , \quad m = 0 , \quad u = 1 , \quad w = 0 , \tag{95}$$

and the mass integrals (92) and (94) yield

$$I = 0 . \tag{96}$$

Therefore, for $\beta = 0$, the constant-section beam and the optimal beam are identical.

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